Switching from stable to unknown unstable steady states of dynamical systems

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(Received 19 March 2008; revised manuscript received 11 April 2008; published 11 August 2008)

We demonstrate that a dynamical system can be switched from a stable steady state to a previously unknown unstable (saddle) steady state using proportional feedback coupling to an auxiliary unstable system. The simplest one-dimensional nonlinear model is treated analytically, the more complicated two-dimensional pendulum is considered numerically, while the damped Duffing-Holmes oscillator is investigated analytically, numerically, and experimentally. Experiments have been performed using a simplified version of the electronic Young-Silva circuit imitating the dynamical behavior of the Duffing-Holmes system. The physical mechanism behind the switching effect is discussed.

DOI: 10.1103/PhysRevE.78.026205 PACS number(s): 05.45.−a, 07.05.Dz, 84.30.Ng

I. INTRODUCTION

Automatic control of dynamical systems is one of the most important fields in engineering science [1]. Although engineers and applied mathematicians have solved many basic problems a long time ago, the pioneering idea of "controlling chaos" introduced by Ott, Grebogi, and Yorke (OGY) [2] inspired many physicists worldwide to develop new techniques for controlling chaos, e.g., [3–8]. The OGY method and new approaches are based on the fact that a chaotic attractor embeds an infinite number of unstable periodic orbits (UPOs) which can be stabilized only through tiny, carefully chosen perturbations.

Chaos control deals mostly with the stabilization of UPOs. However, the problem of stabilizing unstable steady states (USSs) is also of great importance, especially in engineering applications. Classical control methods of nonoscillatory states require as a reference point the coordinates of the USS. In many practical cases the location of the USS is either unknown or it may slowly vary with time because of changes in the ambient conditions. Therefore adaptive, reference-free methods, automatically locating the USS are preferable.

The simplest adaptive technique for stabilizing USSs is based on the derivative controller. A perturbation in the form of a derivative dx/dt derived from an observable $x(t)$ does not change the original system, since it vanishes when the variable $x(t)$ approaches the steady state. This technique works well for originally oscillating systems [9–11]. However, it is not applicable for controlling nonoscillating systems, e.g., moving the system from x_{0S}, an originally stable steady state (SSS) to a USS, because $dx_{0S}/dt=0$.

Another adaptive method for stabilizing USSs employs low- (high-) pass filters in the feedback loop [12–15]. Provided the cutoff frequency of the filter is low enough, the filtered image $v(t)$ of the observable $x(t)$ asymptotically approaches the USS and therefore can be used as a reference point in the proportional feedback. This method has been successfully applied to several experimental systems, including electronic circuits [12,13] and lasers [14,15]. However, it turns out that for a wide class of dynamical systems some newly developed methods [3–5,8,12–15] do not work. If an unstable state, say an UPO, is a torsion-free orbit (or in mathematical language an orbit with an odd number of real positive Floquet exponents), more sophisticated controllers involving *unstable* low-pass filters should be used. The idea of using an auxiliary unstable degree of freedom in the feedback loop was introduced in [16] and has been experimentally verified for stabilizing torsion-free UPOs of autonomous (the van der Pol oscillator) [17] and nonautonomous (the Duffing-Holmes oscillator) [18] dynamical systems. An unstable low-pass filter has been also demonstrated to be useful for stabilization of saddle-type steady states (USSs with an odd number of real positive eigenvalues) in originally oscillating systems [19,20].

In this paper, we demonstrate numerically and experimentally that an unstable low-pass filter can be successfully applied to *nonoscillating* systems (resting in a steady state). Our specific aim is to switch from a stable steady state of a dynamical system to an unknown saddle-type unstable steady state by means of only a small perturbation.

II. PROPORTIONAL FEEDBACK AND STABLE LOW-PASS FILTER

A dynamical system

$$\dot{x} = a(x - x^*)$$

with $a>0$ has an unstable fixed point $x_0=x^*$. If the value of x^* is known, the point x_0 can be stabilized by means of a simple proportional feedback force $k(x^*-x)$:

$$\dot{x} = a(x - x^*) + k(x^* - x) \equiv (a-k)(x - x^*).$$

The steady state of the controlled system is $x_0=x^*$, i.e., exactly the same as that of the free-flowing system. The feedback does not change the location of the steady state, but makes it stable if $k>a$.

If the steady state is unknown one can think of the conventional low-pass tracking filter used to stabilize USSs in chaotically oscillating systems [12–14]. Let us try to apply such a filter to the unstable system by inserting it in the feedback loop:

$$\dot{x} = a(x - \xi) + k(v - x),$$

$$\dot{v} = \omega_f(x - v).$$

Here ξ is an unknown parameter, k is the control gain, v is the variable of the stable filter, and ω_f is its cutoff frequency. However, by means of simple stability analysis one can show that the new fixed point $(x_0, v_0) = (\xi, \xi)$ is a saddle, i.e., the stable controller fails to stabilize USSs of this type for any set of the parameters a, k, and ω_f.

III. UNSTABLE LOW-PASS FILTER

Let us consider a simple one-dimensional nonlinear example,

$$\dot{x} - x - x^2 = \xi.$$

For $\xi < 0.25$ the system has two real steady states $x_{01,02} = -0.5 \mp \sqrt{0.25 - \xi}$. The first one, x_{01}, is a stable and the second one an unstable point. If the value of the parameter ξ is unknown for some reasons or ξ is not a constant, but slowly varies with time, the locations of both fixed points become unspecified. Now we supplement the system with an auxiliary degree of freedom implemented by an unstable first-order filter [19] in the feedback loop:

$$\dot{x} = x + x^2 + \xi + k(u - x),$$

$$\dot{u} = \omega_f(u - x).$$

Here u is the variable of the unstable filter. The steady states of the closed-loop system (x, u) are $(x_{01,02}, x_{01,02})$, i.e., their x components are exactly the same as those of the free-flowing systems. Their stability properties can be easily checked using standard analysis methods. The parameter matrices of the linearized equations are

$$A_{1,2} = \begin{pmatrix} 1 + 2x_{01,02} - k & k \\ -\omega_f & \omega_f \end{pmatrix}.$$

The corresponding traces $\sigma_{1,2}$ and the determinants $\Delta_{1,2}$ of the matrices $A_{1,2}$ are

$$\sigma_{1,2} = \mathrm{Tr}(A_{1,2}) = 1 + 2x_{01,02} - k + \omega_f,$$

$$\Delta_{1,2} = \det(A_{1,2}) = (1 + 2x_{01,02})\omega_f.$$

Since $\Delta_1 < 0$, the first fixed point x_{01}, originally a stable point, becomes a saddle, i.e., an unstable point. The eigenvalues of the characteristic equation

$$\lambda^2 - \sigma_1\lambda + \Delta_1 = 0$$

are both real and have opposite signs. Instability of the fixed point depends neither on the parameter ξ nor on the value of the control gain $k > 0$.

Since $\Delta_2 > 0$, the second, originally unstable, point x_{02} becomes a stable one if $\sigma_2 < 0$. Its stability does not depend on ξ, similarly to the first fixed point, but here the trace σ_2 should be negative, that is, the control gain k should exceed some threshold value ($k > k_{\mathrm{th}} = 1 + 2x_0 + \omega_f$). The eigenvalues $\lambda_{1,2}$ of the corresponding characteristic equation

$$\lambda^2 - \sigma_2\lambda + \Delta_2 = 0$$

either are both real and negative or $\mathrm{Re}\,\lambda_{1,2} < 0$. The stabilized point is either a stable node, if $\sigma_2^2 > 4\Delta_2$, or a stable spiral, if $\sigma_2^2 < 4\Delta_2$.

The fastest control is achieved when both negative eigenvalues are equal: $\lambda_1 = \lambda_2 = \sigma_2/2 < 0$. This is satisfied if the discriminant $D \equiv \sigma_2^2 - 4\Delta_2 = 0$, yielding $k_{\mathrm{opt}} = k_{\mathrm{th}} + 2\sqrt{\Delta_2}$.

In summary, the unstable filter in the feedback loop inverts the stability properties of the two steady points. The originally stable point loses its stability, while the originally unstable point (saddle) gains stability.

IV. PHYSICAL MODELS

In this section we illustrate the performance of unstable controller for two examples, namely, a mechanical pendulum and the Duffing-Holmes system.

A. Switching the states of a mechanical pendulum

As a simple second-order nonlinear example, we consider a mechanical pendulum

$$\ddot{\varphi} + \beta\dot{\varphi} + \sin\varphi = \xi. \tag{1}$$

Here φ is the angle between the downward vertical and the pendulum, β is a damping parameter, and ξ is a constant or slowly varying torque. There are two steady states: $[\varphi_{01}, \dot{\varphi}_{01}] = [\arcsin\xi, 0]$ and $[\varphi_{02}, \dot{\varphi}_{02}] = [\pi - \arcsin\xi, 0]$. The first one is a stable fixed point (either a spiral or a node, depending on the damping β). The second one is an unstable fixed point (a saddle). We rewrite Eq. (1) in a more convenient form, and in order to destabilize the φ_{01} and to stabilize the φ_{02} add to the right-hand side of the equation for the angular velocity ω a feedback torque combined of an observable $\varphi(t)$ and its filtered image $u(t)$, similarly to the previous example:

$$\dot{\varphi} = \omega, \tag{2a}$$

$$\dot{\omega} = -\beta\omega - \sin\varphi + \xi + k(u - \varphi), \tag{2b}$$

$$\dot{u} = \omega_f(u - \varphi). \tag{2c}$$

The switching from the originally stable state to the originally unstable state including the transient process is shown in Fig. 1 for two different values of the damping parameter β. To achieve stability the cutoff frequency of the filter ω_f should be set sufficiently low ($\omega_f < \beta$ [20]). For small β this leads to a very slow transient. For larger β, ω_f can be increased and the transient becomes shorter.

B. Switching the states of the Duffing-Holmes system

The second example is the Duffing-Holmes oscillator. We consider it without external periodic drive but with a constant or slowly varying force ξ:

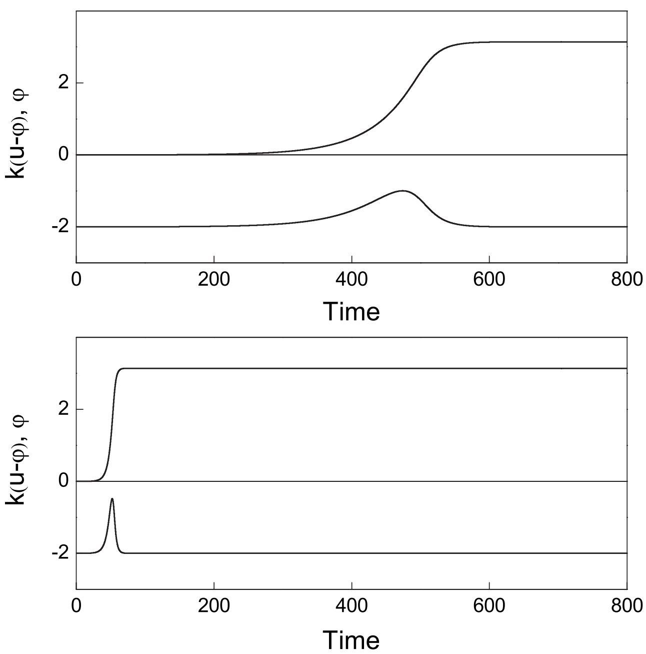

FIG. 1. Switching from the stable to the unstable state of the pendulum from Eq. (2) with $\xi=0$ $(\varphi_{01}=0,\varphi_{02}=\pi)$. Top: $\beta=0.1$, $k=2.1$, $\omega_f=0.05$. Bottom: $\beta=2$, $k=6$, $\omega_f=1$. Upper traces in the top and bottom plots show the main observable φ; lower traces (shifted down by 2 for clarity) show the control term $k(u-\varphi)$. Control is turned on at $t=0$.

$$\ddot{x}+b\dot{x}-x+x^3=\xi. \tag{3}$$

Here b is the damping coefficient. In the case where the biasing force ξ is not too large ($|\xi|<2/\sqrt{27}$) Eq. (3) has three real steady state solutions $(x_{01,02,03},\dot{x}_{01,02,03})=(x_{01,02,03},0)$. Their x projections are found from a cubic algebraic equation ${x_0}^3-x_0-\xi=0$:

$$x_{01,02}=-h\cos\frac{\pi\mp\theta}{3},\quad x_{03}=h\cos\frac{\theta}{3},$$

with $h=2/\sqrt{3}$ and $\theta=\arccos(3\xi/h)$. In the limiting case of zero bias ($\xi=0$), the auxiliary angle $\theta=\pi/2$ and $x_{01}^*=-1$, $x_{02}^*=0$, $x_{03}^*=1$, as expected, and correspond to the symmetric double-well potential of the Duffing-Holmes system. Two of the steady states $(x_{01},0)$ and $(x_{03},0)$ are stable fixed points, while the intermediate point $(x_{02},0)$ is a saddle. We add a feedback that combines the observable $x(t)$ and the output of the filter $u(t)$:

$$\dot{x}=y, \tag{4a}$$

$$\dot{y}=-by+x-x^3+\xi+k(u-x), \tag{4b}$$

$$\dot{u}=\omega_f(u-x). \tag{4c}$$

Let us consider the steady states and their stability properties for $k>0$. The x and the y projections remain unchanged, while the u projection coincides with the x projection. This means that control does not influence the location of the steady states. However, the originally unstable state $(x_{02},0)$ under certain conditions can become a stable steady state $(x_{02},0,x_{02})$. To check the stability of the system, we linearize Eq. (4) around this point:

$$\dot{x}=y, \tag{5a}$$

$$\dot{y}=-by-(k-1+3x_{02}^2)x+ku, \tag{5b}$$

$$\dot{u}=\omega_f(u-x), \tag{5c}$$

and analyze its characteristic equation

$$\lambda^3+(b-\omega_f)\lambda^2+(k-1+3x_{02}^2-b\omega_f)\lambda+\omega_f=0. \tag{6}$$

The system is stable if the real parts of all three eigenvalues of Eq. (6) are negative. The necessary and sufficient conditions can be found using the Routh-Hurwitz matrix

$$H=\begin{pmatrix} b-\omega_f & \omega_f & 0 \\ 1 & k-1-b\omega_f+3x_{02}^2 & 0 \\ 0 & b-\omega_f & \omega_f \end{pmatrix}.$$

The eigenvalues $\mathrm{Re}\,\lambda_{1,2,3}$ are all negative if the diagonal minors of the H matrix are all positive:

$$\Delta_1=b-\omega_f>0, \tag{7a}$$

$$\Delta_2=(b-\omega_f)(k-1-b\omega_f+3x_{02}^2)-\omega_f>0, \tag{7b}$$

$$\Delta_3=\omega_f\Delta_2>0. \tag{7c}$$

These inequalities are satisfied if

$$0<\omega_f<b, \tag{8a}$$

$$k>k_{\mathrm{th}}=\frac{b}{b-\omega_f}+b\omega_f-3x_{02}^2. \tag{8b}$$

For small b and ξ the threshold gain $k_{\mathrm{th}}\approx b/(b-\omega_f)$. For example, at $b=0.1$, $\omega_f=0.03$, $\xi=0$, the gain $k_{\mathrm{th}}\approx1.43$. In order to find the optimal value of the gain corresponding to the maximum convergence rate we have solved Eq. (6) numerically (Fig. 2). $\mathrm{Re}\,\lambda<0$ and $|\mathrm{Re}\,\lambda|$ have maximal values at $k_{\mathrm{opt}}\approx2.3$. We note that in Fig. 2 two originally positive λ become negative at $k\approx1.43$, coinciding well with the k_{th} found from the Routh-Hurwitz criteria.

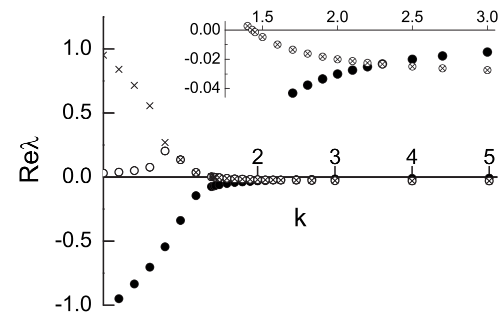

FIG. 2. Dependence of the real part of the eigenvalues Re λ on the control gain k from Eq. (6). $\xi=0$, $b=0.1$, $\omega_f=0.03$.

Numerical results obtained by integrating Eq. (4) are presented in Fig. 3. The lengths of transient processes in the

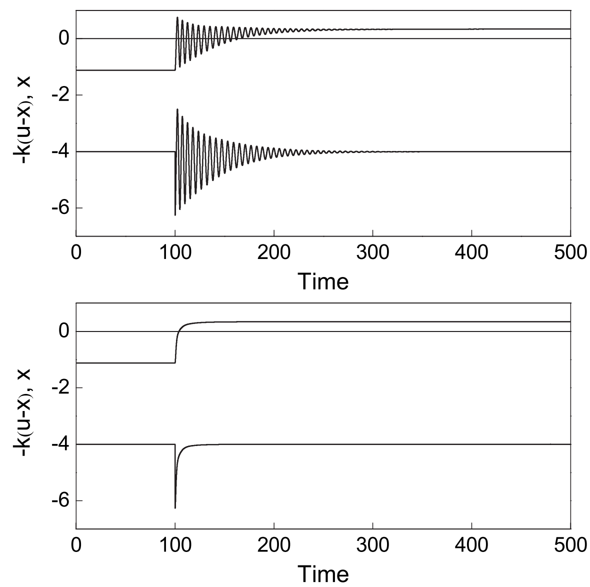

FIG. 3. Switching from the stable to the unstable state of the Duffing-Holmes oscillator from Eq. (4) with $k=2$ and $\xi=-0.3$. Top: $b=0.1$, $\omega_f=0.03$. Bottom: $b=2$, $\omega_f=0.1$. Upper traces in the top and bottom plots show the main observable x; lower traces (shifted down by 4 for clarity) show the control term $k(u-x)$. Control is turned on at $t=100$.

Duffing-Holmes system, as in the pendulum, are different for low and high damping.

V. EXPERIMENTAL SETUP AND RESULTS

An electronic circuit imitating dynamics of the Duffing-Holmes oscillator (Fig. 4) is composed of the operational amplifier OA1, the elements $R1-R3$, R, L, C, the diodes $D1$, and $D2$. Actually it is a simplified version of the Young-Silva oscillator [21], used to demonstrate stabilization of a torsion-free UPO [18]. The rest of the circuit is a controller. The

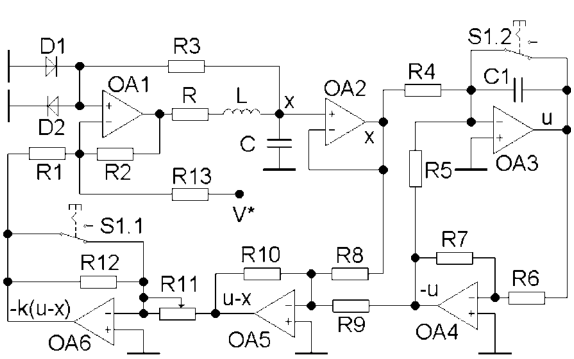

FIG. 4. Duffing-Holmes circuit with an unstable controller in the feedback loop. $R_1-R_{12}=10$ $k\Omega$, $R_{13}=1$ MΩ, $L=19$ mH, $C=470$ nF. OA1–OA6, LM741 integrated circuits; D1 and D2, 1N4148 diodes. V^* is a dc voltage source. The corresponding bias voltage applied to the oscillator $V_0=-V^*R_2/R_{13}$. S1.1-S1.2 is an electronically controlled double switch. The position of the switch corresponds to the controller in the off state. R and C_1 are specified in the caption to Fig. 5.

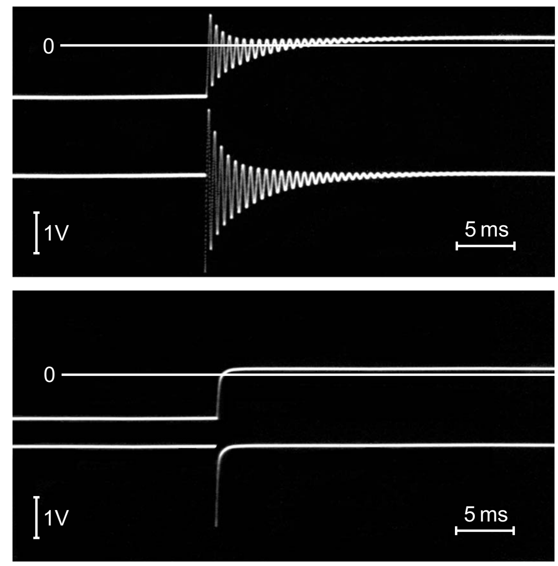

FIG. 5. Experimental control of the steady states of the Duffing-Holmes electrical circuit shown in Fig. 4 with $R_{11}=5$ kΩ ($k=2$) and $V^*=30$ V ($V_0=-300$ mV). (Top) $R=20$ Ω ($b=0.1$), $C_1=330$ nF ($\omega_f=0.03$); (bottom) $R=400$ Ω ($b=2$), $C_1=100$ nF ($\omega_f=0.1$). Upper traces in the top and bottom photos show the main signal $V_C \propto x$; the lower traces represent the control signal $V_{\text{contr}} \propto -k(u-x)$ taken from the OA6.

OA2 stage is a buffer, the OA3 and OA4 stages are an unstable first-order low-pass filter, the OA5 stage is an inverting adder, and finally the OA6 stage is an inverting amplifier used to set the control gain $k=R_{12}/R_{11}$. Location of the unstable steady state can be varied by means of the external voltage source V^*. The experimental results are presented in Fig. 5.

The main difference between the results in Figs. 1, 3, and 5 and the investigations in [19,20] lies in the following. The electrochemical oscillator [19], the pendulum, and the Lorenz system [20] are all in the oscillating regimes before the control is turned on. While the control in Figs. 1, 3, and 5 is activated when the systems are fixed in stable steady states (either stable spirals or stable nodes). Although it is stated nowhere in the text of [19,20], the presented illustrations of originally oscillating and rotating systems give an inadequate impression that a saddle point can be stabilized only if it is surrounded or approached by the trajectories of the limit cycles and chaotic attractors. In our case the original SSS and the USS are fixed and rather remote objects in the phase space. In addition, the examples of overdamped systems (bottom plots in Figs. 1, 3, and 5) show that in order to switch from a SSS to a saddle it is not necessary even for the transient trajectories to oscillate around the USS. In contrast, the target can be reached point blank.

VI. CONCLUSIONS

We have demonstrated both numerically and experimentally the efficiency of an unstable filter in switching a dynamical system from stable steady states to unknown unstable steady states (saddles). Thus we have extended the field of application of unstable controllers from oscillatory to nonoscillatory dynamical systems. The controller automatically locates the unknown unstable fixed point and uses it as a reference point in the feedback loop. The basic mechanism behind the switching effect is the simultaneous inversion of the stability properties of the states: originally stable nodes and spirals lose their stability and become unstable points (saddles) and, vice versa, the originally unstable points (saddles) gain stability and become stable nodes or spirals.

ACKNOWLEDGMENTS

We are grateful to V. Pyragas for discussion of the systems investigated in [19,20] and to T. Pyragienė for drawing our attention to the Routh-Hurwitz stability criteria. E.T. was partially supported by the Gifted Student Fund, Lithuania.

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